ILLAWARRA TUTORING

This worksheet explores key properties of functions: continuity, increasing/decreasing behaviour, and symmetry. You will answer questions that test your understanding of these characteristics and how they affect the behaviour of functions.

Easy Questions

1. Determine whether the function $f(x) = \dfrac{1}{x-2}$ is continuous at $x = 2$.
2. Define what it means for a function to be increasing on an interval.
3. Define what it means for a function to be decreasing on an interval.
4. Determine whether the function $f(x) = x^2$ is symmetric about the y-axis, the origin, or neither.
5. Decide if the function $f(x) = |x|$ is continuous for all real numbers and state its symmetry property.

Intermediate Questions

6. State the definition of continuity of a function at a point c.
7. Explain the Intermediate Value Property for continuous functions and provide an example.
8. Consider the function
$$f(x) = \begin{cases} x+1 & \text{if } x < 0, \\ x^2 & \text{if } x \geq 0. \end{cases}$$
Determine all points where f is continuous.
9. Sketch a diagram of a function that has a removable discontinuity. Label the point of discontinuity.
10. Consider the function $f(x) = x^2 - 4$. Identify the intervals on which f is decreasing and those on which it is increasing.
11. Determine whether the function $f(x) = x^3 - x$ is symmetric about the y-axis, the origin, or neither.
12. Explain the difference between a removable discontinuity and a jump discontinuity.

13. Given the function
$$f(x) = \frac{x^2 - 1}{x - 1} \quad \text{for } x \neq 1,$$
state whether this function can be redefined at $x = 1$ to be continuous, and explain why.

14. State the definition of an even function and provide an example of an even function that is continuous for all x.

15. State the definition of an odd function and provide an example of an odd function that is continuous for all x.

16. Describe in your own words how you can determine if a function is increasing or decreasing on an interval without using calculus techniques.

17. Consider the function
$$f(x) = \begin{cases} 2x + 3 & \text{if } x < 1, \\ x^2 + 1 & \text{if } x \geq 1. \end{cases}$$
Determine whether f is continuous at $x = 1$ and comment on its behaviour around that point.

18. Using a table of values, explain how one might identify whether a function is increasing, decreasing, or constant on an interval.

19. Provide an example of a function that is continuous but not monotonic (i.e. neither entirely increasing nor entirely decreasing) on its domain.

20. Describe what is meant by a function being symmetric and explain why recognising symmetry can simplify the study of a function's properties.

Hard Questions

21. Let
$$f(x) = \begin{cases} \frac{x^2-4}{x-2} & \text{if } x \neq 2, \\ k & \text{if } x = 2. \end{cases}$$
Determine the value of k for which f is continuous at $x = 2$, and explain your reasoning.

22. Prove that if a function f is even and continuous on $\mathbb{R}$, then its graph is symmetric with respect to the y-axis.

23. Illustrate using a diagram how a jump discontinuity appears on the graph of a function.

24. Consider the function
$$f(x) = \begin{cases} x - 1 & \text{if } x < 2, \\ 3 - x & \text{if } x \geq 2. \end{cases}$$
Determine whether f is continuous at $x = 2$ and explain your reasoning.

25. Construct a continuous function that is neither entirely increasing nor entirely decreasing. Describe the key features of your function.

26. Let
$$f(x) = \frac{x^2 - 9}{x - 3} \quad \text{for } x \neq 3.$$
Determine if f can be made continuous at $x = 3$ by redefining $f(3)$, and explain your answer.

27. Consider a function f that is continuous on the interval $[a, b]$ and suppose that $f(a)$ and $f(b)$ have opposite signs. Explain how this information guarantees that f has a zero in the interval (a, b).

28. Given that f is an odd function and continuous on $\mathbb{R}$, prove that $f(0) = 0$.

29. Let f be a function defined on $\mathbb{R}$ such that $f(-x) = f(x)$ (i.e. f is even) and assume f is continuous. If $f(1) = 2$, what can you say about $f(-1)$? Explain your answer.

30. Provide an example of a function that is continuous everywhere but exhibits different behaviours (increasing on some intervals and decreasing on others). Identify these intervals and justify your answer.